J. Fluid Mech. (1973), *vol.* 61, *part* 2, *pp.* 305–321
Printed in Great Britain

A perturbation analysis of the laminar far wake behind a symmetrical two-dimensional body in a uniform shear flow

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(Received 18 April 1973)

An aspect of the laminar far wake behind a symmetrical two-dimensional body placed in a uniform shear flow is described theoretically by means of the Oseen type of successive approximation, in which the shear is regarded as a small perturbation on a uniform stream. The expression for the stream function is determined up to the third approximation both in and outside the wake region, and the region in which the results of the perturbation analysis are valid is also determined. The stream function is found to contain four constants which cannot be determined from the boundary conditions for the far wake. The analysis also shows that the spreading of the wake is greater towards the side of smaller velocity than the side of larger velocity, the asymmetrical feature of the velocity defect becoming more evident as the distance from the obstacle is increased: the point which shows the maximum velocity defect shifts to the low-velocity side.

1. Introduction

The laminar flow field at large distances from a finite body located in a uniform flow of an incompressible viscous fluid is of basic importance and has been theoretically treated by many investigators. Imai (1951), among others, made a detailed analysis of the two-dimensional far flow field for an arbitrary Reynolds number, based on the Oseen type of approximation of the Navier–Stokes equations. The same problem was considered by Chang (1961) by means of matched asymptotic expansions of the co-ordinate type. Childress (1961) subsequently considered axially symmetric and general three-dimensional far flow fields by the same method.

There are many examples in practice, however, where a parallel flow of fluid is non-uniform to the extent that velocity varies in magnitude across the stream. Such a flow exhibits shear characteristics, a typical example being the flow in the presence of a plane solid boundary, where the velocity increases with distance from the boundary. In an attempt to obtain the low Reynolds number flow around a circular cylinder placed in a uniform shear flow, Bretherton (1962) considered the two-dimensional flow at large distances from the cylinder. Hunt (1971) obtained a solution for the laminar wake far downstream of a cylindrical body placed with its generators perpendicular to the flow on a surface above which there is a simple shear flow described by $u = Gy$, where y is the distance normal to the surface, G is a constant velocity gradient and u is the velocity component along the surface. An important difference between the velocity in this wake and in those behind bodies in a uniform flow is that here the perturbation velocity decreases with distance downstream (say, x) in proportion to x^{-1} as opposed to $x^{-\frac{1}{2}}$ in the latter case. In the solutions of Bretherton and Hunt the velocity profile in the wakes is described by the similarity variable $y/x^{\frac{1}{3}}$ as opposed to $y/x^{\frac{1}{2}}$ in the case of a uniform flow. In this connexion, it should be remarked that their analyses are deliberately restricted to the case where the shear is the dominant feature of the flow. The wakes are fundamentally different from those behind bodies in a uniform stream, which cannot be deduced from their results as a special case.

In reality, however, there will be a lot of cases or regions of flow in which the effect of shear on the development of wakes can be interpreted as a small perturbation to the wakes behind bodies placed in a uniform stream. A theoretical analysis of such wakes is the subject of the present paper and will correspond to an opposite extreme of the cases which were treated by Bretherton and Hunt. The present work has resulted from the authors' interest in the general problem of flows around bodies located in non-uniform oncoming streams. It should be remarked here that Kawaguti (1956) investigated the low Reynolds number flow around a circular cylinder located in a uniform shear flow by regarding the shear as a perturbation on a uniform stream. However, characteristics of the wake behind such a cylinder were not considered in Kawaguti's paper at all. There will necessarily exist an upper limit of the range of x where the solutions of wakes obtained by the perturbation analysis are valid. The limit will be considered later.

For the sake of simplicity, a two-dimensional symmetrical body aligned parallel to the uniform shear flow is assumed in the present analysis.

Finally, it may be noted that the motion in the wake at the large Reynolds number, which is the case of practical importance, is unstable with respect to infinitesimal disturbances, because the velocity profile in the wake has points of inflexion. For a slender streamlined body, the flow in the wake, even if steady just behind the body, becomes turbulent further downstream; whilst for a bluff body the flow in the wake is definitely not steady but quasi-periodic or turbulent when the Reynolds number is large. Therefore, it may be worth mentioning that there exists a similarity between the mean velocity profiles of the free turbulent shear flows such as wakes or jets and those of the corresponding laminar ones. The turbulent eddy viscosities for free turbulent shear flows in general have been found to be proportional to the product of the maximum velocity difference in the shear layer and its width. Since both are functions of the streamwise coordinate x, the turbulent eddy viscosities also become functions of x alone. The mean flow pattern of turbulent shear flows of such an eddy viscosity could be investigated as an extension of the laminar theory. Moreover, a fundamental understanding of the equivalent laminar flow problem is always necessary before embarking on a phenomenological analysis of a turbulent flow.

2. Basic equations

We introduce a Cartesian co-ordinate system (x, y), where x is the streamwise distance measured from an appropriate point near the obstacle and y is the normal distance measured from the x axis; the corresponding velocities will be denoted by u and v, respectively. Motions of the fluid are governed by the Navier–Stokes equations and the equation of continuity. For incompressible flow, when the stream function Ψ is introduced by the usual definition

$$\partial\Psi/\partial y = u, \quad \partial\Psi/\partial x = -v, \tag{2.1}$$

the equation of continuity is automatically satisfied. The Navier–Stokes equations for the two-dimensional steady flow of an incompressible viscous fluid are then written in the form

$$\frac{\partial\Psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\omega}{\partial y} = \nu\Delta\omega, \tag{2.2}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, ν is the kinematic viscosity and ω is the vorticity, defined by

$$\omega = \partial v/\partial x - \partial u/\partial y = -\Delta\Psi. \tag{2.3}$$

At infinitely large distances, the components of the fluid velocity are assumed to be

$$u_\infty = U + Gy, \quad v_\infty = 0, \tag{2.4}$$

where U and G are constants. Then, the stream function Ψ_∞ corresponding to (2.4) becomes

$$\Psi_\infty = \int u_\infty dy = Uy + \tfrac{1}{2}Gy^2. \tag{2.5}$$

If we write

$$\Psi = \Psi_\infty + \psi, \tag{2.6}$$

we have from (2.1)

$$u = U + Gy + \partial\psi/\partial y, \tag{2.7}$$

$$v = -\partial\psi/\partial x, \tag{2.8}$$

so that $\partial\psi/\partial x$ and $\partial\psi/\partial y$ will be small at large distances from the obstacle. Thus, we can write (2.3) and (2.2) in the forms

$$\omega = -(G + \Delta\psi), \tag{2.9}$$

$$\Delta^2\psi - 2k(1+\alpha y)\frac{\partial\Delta\psi}{\partial x} = \frac{2k}{U}\left(\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}\right), \tag{2.10}$$

where $2k = U/\nu$ and $\alpha = G/U$. The form of (2.10) makes it plausible to assume the following forms for ψ and ω:

$$\psi = U(\psi_1 + \alpha\psi_2) + o(\alpha), \tag{2.11}$$

$$\omega = U\{\omega_1 + \alpha(\omega_2 - 1)\} + o(\alpha), \tag{2.12}$$

where

$$\omega_1 = -\Delta\psi_1, \quad \omega_2 = -\Delta\psi_2. \tag{2.13}$$

Substituting (2.11) into (2.9) and (2.10) and comparing like powers of α, we obtain

$$\Delta\omega_1 - 2k\frac{\partial\omega_1}{\partial x} = 2k\left(\frac{\partial\psi_1}{\partial y}\frac{\partial\omega_1}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial\omega_1}{\partial y}\right), \tag{2.14}$$

$$\Delta\omega_2 - 2k\frac{\partial\omega_2}{\partial x} - 2ky\frac{\partial\omega_1}{\partial y} = 2k\left(\frac{\partial\psi_1}{\partial y}\frac{\partial\omega_2}{\partial x} + \frac{\partial\psi_2}{\partial y}\frac{\partial\omega_1}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial\omega_2}{\partial y} - \frac{\partial\psi_2}{\partial x}\frac{\partial\omega_1}{\partial y}\right). \tag{2.15}$$

Since the vorticity in the original inviscid stream, which is $-G$, is constant, Kelvin's law of vorticity requires that the flow outside the wake region be governed by

$$\Delta\psi_1 = \Delta\psi_2 = 0. \qquad (2.16), (2.17)$$

Therefore, (2.14) and (2.15) are understood to form the fundamental equations for the flow in the wake region, while (2.16) and (2.17) show that the flow field outside the wake region can be determined by means of potential flow theory. Some constants which will appear in the solutions can be obtained by the matching conditions at the edge of the wake. The matching conditions and the boundary conditions will be discussed in the course of the following analysis.

Since ψ_1 represents the solution for $\alpha = 0$, it should be understood to describe the wakes behind obstacles located in a uniform stream. Therefore, ψ_2 gives the first-order effect of the main-stream shear on the flow in wakes.

3. Solution for ψ_1

The laminar far wake behind any symmetrical obstacle in a uniform stream, which is described by ψ_1, has been well established by previous investigators. Here the results of Imai (1951) will briefly be summarized for later reference, because use must be made of ψ_1 in order to determine ψ_2.

For sufficiently large x, $\partial\psi_1/\partial x$ and $\partial\psi_1/\partial y$ are small compared with U and, therefore, the first approximation to ψ_1 or ω_1 can be calculated by neglecting the terms in (2.14) that are quadratic in $\partial\psi_1/\partial x$ and $\partial\psi_1/\partial y$. If the first approximation to ω_1 is denoted by $\omega_1^{(1)}$, $\omega_1^{(1)}$ now satisfies the equation

$$\Delta\omega_1^{(1)} - 2k\,\partial\omega_1^{(1)}/\partial x = 0. \tag{3.1}$$

Introducing the complex variables

$$z = x+iy, \quad \zeta = \xi + i\eta,$$

related by the equation

$$2kz = \zeta^2, \tag{3.2}$$

Imai obtained the solution of (3.1) in the form

$$\omega_1^{(1)} = -\Delta\psi_1^{(1)} = -\mathscr{R}\left\{\frac{2k^2 m}{\pi^{\frac{1}{2}}U}\frac{e^{-\eta^2}}{\zeta}\right\}. \tag{3.3}$$

Here $\mathscr{R}$ means 'the real part of ' and m is a constant related to the fluid density ρ and the drag D of the obstacle by the relation

$$m = D/\rho U. \tag{3.4}$$

Equation (3.3) now yields

$$\psi_1^{(1)} = -\frac{m}{2U}\left(\operatorname{erf}\eta - \frac{\theta}{\pi}\right) \quad (-\pi \leqslant \theta \leqslant \pi), \tag{3.5}$$

where

$$\theta = \arg z, \quad \operatorname{erf}\eta = \frac{2}{\pi^{\frac{1}{2}}}\int_0^\eta e^{-t^2}\,dt.$$

Equation (3.5) is valid throughout the flow region at a considerable distance from the obstacle, both in and outside the wake. Especially, it should be remarked that, in the region outside the wake, $\psi_1^{(1)}$ takes the form of

$$\psi_1^{(1)} = \mathscr{I}\{(m/2\pi U)\log z\},$$

in which $\mathscr{I}$ stands for 'the imaginary part of'.

By applying an iteration procedure starting with $\psi_1^{(1)}$, Imai obtained the higher order approximations in the following forms:

$$\psi_1 = \psi_1^{(1)} + \psi_1^{(2)} + \psi_1^{(3)} + \ldots, \tag{3.6}$$

with

$$\psi_1^{(2)} = -\frac{km^2}{4\pi^{\frac{1}{2}}U^2}\frac{1}{\xi}\left(2^{\frac{1}{2}}\operatorname{erf}2^{\frac{1}{2}}\eta - e^{-\eta^2}\operatorname{erf}\eta\right), \tag{3.7}$$

$$\begin{aligned}\psi_1^{(3)} = {} & \frac{km^2}{\pi^{\frac{3}{2}}U^2}\frac{1}{\xi^2}\left[\pi^{\frac{1}{2}}\operatorname{erf}\eta + (\log\xi)\,\eta\, e^{-\eta^2}\right] \\ & + \frac{km^3}{48\pi U^3}\frac{1}{\xi^2}\{4\times 3^{\frac{1}{2}}\eta\, e^{-\eta^2}R(\eta) - 14\times 3^{\frac{1}{2}}(\operatorname{erf}3^{\frac{1}{2}}\eta - \operatorname{erf}\eta) \\ & + 12\times 2^{\frac{1}{2}}e^{-\eta^2}\operatorname{erf}2^{\frac{1}{2}}\eta + 6\,e^{-2\eta^2}\operatorname{erf}\eta + 3\,\pi^{\frac{1}{2}}\eta\, e^{-\eta^2}(\operatorname{erf}\eta)^2 - 12\times 3^{\frac{1}{2}}\operatorname{erf}\eta \\ & - 4\times 3^{\frac{1}{2}}\pi^{-\frac{1}{2}}(\log\xi)\,\eta\, e^{-\eta^2}\} + \frac{B}{U}\frac{1}{\xi^2}\eta\, e^{-\eta^2},\end{aligned} \tag{3.8}$$

$$R(\eta) = \int_0^\eta (\operatorname{erf}3^{\frac{1}{2}}\eta - \operatorname{erf}\eta)\, e^{\eta^2}\,d\eta,$$

in the wake, and

$$\psi_1 = \mathscr{I}\{f(z)\}, \tag{3.9}$$

with

$$f(z) = \frac{m}{2\pi U}\log z - i\,\frac{k^{\frac{1}{2}}m^2}{4\pi^{\frac{1}{2}}U^2}\frac{1}{z^{\frac{1}{2}}} + \left(\frac{3^{\frac{1}{2}}km^3}{8\pi^2U^3} - \frac{m^2}{2\pi^2U^2}\right)\frac{\log z}{z} + \frac{a}{z} + \ldots, \tag{3.10}$$

for the region outside the wake. Here, B and a are real constants which cannot be determined from any boundary condition on the far wake. These constants have been believed to be connected in some way with the flow near the obstacle, which is neglected in the far-wake analysis (Stewartson 1957).

4. Solutions for ψ_2

4.1. *First approximation*

When x is large, the terms on the right-hand side of (2.15) that are quadratic in small quantities can be neglected for the first approximation $\omega_2^{(1)}$. Then we have

$$\left.\begin{aligned}\Delta\omega_2^{(1)} - 2k\,\partial\omega_2^{(1)}/\partial x &= 2ky\,\partial\omega_1^{(1)}/\partial x,\\ \text{or} \quad \Delta(\omega_2^{(1)} + 2k\,\partial\psi_2^{(1)}/\partial x) &= 2ky\,\partial\omega_1^{(1)}/\partial x.\end{aligned}\right\} \tag{4.1}$$

Remembering (3.2), we have the relations

$$\frac{\partial}{\partial x} = \frac{k}{\xi^2+\eta^2}\left(\xi\frac{\partial}{\partial \xi} - \eta\frac{\partial}{\partial \eta}\right),$$

$$\frac{\partial}{\partial y} = \frac{k}{\xi^2+\eta^2}\left(\xi\frac{\partial}{\partial \eta} - \eta\frac{\partial}{\partial \xi}\right)$$

and

$$\Delta = [k^2/(\xi^2+\eta^2)]\,\Delta^*,$$

where

$$\Delta^* = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2.$$

Substitution of (3.3) into the right-hand side of (4.1), together with the above differentiation formulae, yields

$$\Delta^*\left(\omega_2^{(1)} + 2k\frac{\partial \psi_2^{(1)}}{\partial x}\right) = -\frac{4km}{\pi^{\frac{1}{2}}U}\frac{1}{\xi}(2\eta^4 - 3\eta^2)\,e^{-\eta^2} + O(\xi^{-3}). \tag{4.2}$$

Since the operation $\partial/\partial\xi$ is equivalent in order of magnitude to multiplication by $1/\xi$, we can integrate (4.2) twice with respect to η to obtain

$$\omega_2^{(1)} + 2k\frac{\partial \psi_2^{(1)}}{\partial x} = \frac{4km}{\pi^{\frac{1}{2}}U}\frac{1}{\xi}\iint (3\eta^2 - 2\eta^4)\,e^{-\eta^2}\,d\eta\,d\eta + O(\xi^{-3})$$

$$= -\frac{2km}{\pi^{\frac{1}{2}}U}\frac{1}{\xi}(\eta^2+1)\,e^{-\eta^2} + O(\xi^{-3}).$$

On using the relation $\omega_2^{(1)} = -\Delta\psi_2^{(1)}$, we have

$$D\psi_2^{(1)} = (2m/\pi^{\frac{1}{2}}kU)\,\xi(\eta^2+1)\,e^{-\eta^2} + O(\xi^{-1}), \tag{4.3}$$

in which

$$D = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} - 2\xi\frac{\partial}{\partial\xi} + 2\eta\frac{\partial}{\partial\eta}. \tag{4.4}$$

Assume the solution of (4.3) to be of the form

$$\psi_2^{(1)} = \xi\eta g_1(\eta) + O(\xi^{-1}), \tag{4.5}$$

where $g_1(\eta)$ is a function of η alone. Substituting (4.5) into (4.3) we have

$$g_1'' + 2\left(\eta + \frac{1}{\eta}\right)g_1' = \frac{2m}{\pi^{\frac{1}{2}}kU}\left(\eta + \frac{1}{\eta}\right)e^{-\eta^2}, \tag{4.6}$$

the prime implying the differentiation with respect to η. The general solution of (4.6) is

$$g_1(\eta) = \frac{m}{\pi^{\frac{1}{2}}kU}\frac{1}{\eta}g_{11}(\eta) - C_1\frac{1}{\eta}g_{12}(\eta) + C_2,$$

in which C_1 and C_2 are constants of integration and

$$g_{11}(\eta) = -\tfrac{1}{4}\eta^2 e^{-\eta^2} + \tfrac{5}{8}\pi^{\frac{1}{2}}\eta\,\mathrm{erf}\,\eta,$$

$$g_{12}(\eta) = e^{-\eta^2} + \pi^{\frac{1}{2}}\eta\,\mathrm{erf}\,\eta.$$

Therefore, the solution of $\psi_2^{(1)}$ becomes

$$\psi_2^{(1)} = \xi\{(m/\pi^{\frac{1}{2}}kU)\,g_{11}(\eta) - C_1 g_{12}(\eta) + C_2\eta\}. \tag{4.7}$$

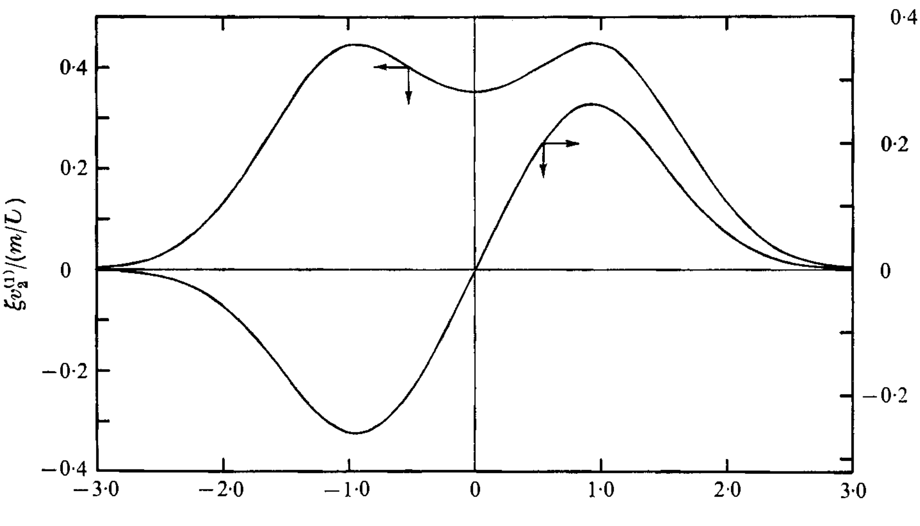

FIGURE 1. Velocity components in the first approximation.

Since $g_{11} \to \pm \frac{5}{8}\pi^{\frac{1}{2}}\eta$ and $g_{12} \to \pm \pi^{\frac{1}{2}}\eta$ as $\eta \to \pm \infty$, we have from (4.7)

$$\psi_2^{(1)} = \xi\eta\{C_2 \pm (5m/8kU - \pi^{\frac{1}{2}}C_1)\}. \tag{4.8}$$

It should be noted here that the flow outside the wake region must be matched to (4.8) at the edge of the wake. Remembering that (3.2) gives

$$2ky = 2\xi\eta, \quad 2kx = \xi^2 - \eta^2,$$

we can obtain the flow outside the wake region which matches to (4.8) in the form

$$\psi_2^{(1)} = \mathscr{I}\{W_2^{(1)}\}, \tag{4.9}$$

where
$$W_2^{(1)} = \{C_2 \pm (5m/8kU - \pi^{\frac{1}{2}}C_1)\}kz, \tag{4.10}$$

the $\pm$ signs standing for $\mathscr{I}(z) \gtrless 0$. Equation (4.10) yields the constant velocity components

$$dW_2^{(1)}/dz = U_2^{(1)} - iV_2^{(1)} = k\{C_2 \pm (5m/8kU - \pi^{\frac{1}{2}}C_1)\},$$

throughout the whole region outside the wake. Since non-zero velocity components at $y \to \pm\infty$ which are induced by the displacement effect of the wake are unacceptable for physical reasons, we must put

$$C_1 = 5m/8\pi^{\frac{1}{2}}kU, \quad C_2 = 0.$$

Therefore, $\psi_2^{(1)}$ can be determined as

$$\psi_2^{(1)} = -(m/4\pi^{\frac{1}{2}}kU)\,\xi(\eta^2 + \tfrac{5}{2})\,e^{-\eta^2} + O(\xi^{-1}), \tag{4.11}$$

which yields the velocity components

$$u_2^{(1)} = \partial\psi_2^{(1)}/\partial y = (m/4\pi^{\frac{1}{2}}U)\,\eta(2\eta^2+3)\,e^{-\eta^2} + O(\xi^{-2}), \tag{4.12}$$

$$v_2^{(1)} = -\partial\psi_2^{(1)}/\partial x = \frac{m}{8\pi^{\frac{1}{2}}U}\frac{1}{\xi}(4\eta^4 + 8\eta^2 + 5)\,e^{-\eta^2} + O(\xi^{-3}). \tag{4.13}$$

The profiles of $u_2^{(1)}$ and $v_2^{(1)}$ are plotted in figure 1. It may be noted that $u_2^{(1)}$ is antisymmetric with respect to the η axis.

4.2. *Second approximation*

To find the second approximation, we write

$$\psi_2 = \psi_2^{(1)} + \psi_2^{(2)}. \tag{4.14}$$

Then, $\psi_2^{(2)}$ satisfies the equation

$$\Delta\left(\omega_2^{(2)} + 2k\frac{\partial\psi_2^{(2)}}{\partial x}\right) = 2k\left(y\frac{\partial\omega_1^{(2)}}{\partial x} + \frac{\partial\psi_1^{(1)}}{\partial y}\frac{\partial\omega_2^{(1)}}{\partial x} + \frac{\partial\psi_2^{(1)}}{\partial y}\frac{\partial\omega_1^{(1)}}{\partial x} - \frac{\partial\psi_1^{(1)}}{\partial x}\frac{\partial\omega_2^{(1)}}{\partial y} - \frac{\partial\psi_2^{(1)}}{\partial x}\frac{\partial\omega_1^{(1)}}{\partial y}\right). \tag{4.15}$$

Substituting (3.5), (3.7) and (4.11) into the right-hand side of (4.15) and rearranging the terms, we obtain

$$\begin{aligned}\Delta^*\left(\omega_2^{(2)} + 2k\frac{\partial\psi_2^{(2)}}{\partial x}\right) = &-\frac{k^2m^2}{2\pi^{\frac{1}{2}}U^2}\xi^{-2}(3\eta X'' + \eta^2 X''') + \frac{k^2m^2}{4\pi U^2}\xi^{-2}\,Y(Z'' + \eta Z''')\\ &+\frac{k^2m^2}{4\pi U^2}\xi^{-2}Z'(2Y' + \eta Y'') - \frac{k^2m^2}{4\pi U^2}\xi^{-2}\eta Y Z'''\\ &+\left\{\frac{k^2m^2}{4\pi^{\frac{3}{2}}U^2}\,\xi^{-3}(\eta Z''' - Z'')\right\} + O(\xi^{-4}),\end{aligned} \tag{4.16}$$

where

$$X = 2^{\frac{1}{2}}\operatorname{erf} 2^{\frac{1}{2}}\eta - e^{-\eta^2}\operatorname{erf}\eta,$$

$$Y = e^{-\eta^2}, \quad Z = (2\eta^2 + 5)\,e^{-\eta^2}.$$

The term in curly brackets in (4.16), which is of order ξ^{-3}, should be considered in the third approximation. The terms of order ξ^{-2} will then be retained in the second approximation. Integrating (4.16) twice with respect to η, we obtain

$$\omega_2^{(2)} + 2k\,\partial\psi_2^{(2)}/\partial x = (k^2m^2/4\pi U^2)\,\xi^{-2}\,Q(\eta) + O(\xi^{-3}), \tag{4.17}$$

in which

$$Q(\eta) = (5 - 2\eta^2)\,e^{-2\eta^2} - 2\pi^{\frac{1}{2}}(2\eta^3 + \eta)\,e^{-\eta^2}\operatorname{erf}\eta + 2(2\pi)^{\frac{1}{2}}\,\eta\operatorname{erf} 2^{\frac{1}{2}}\eta.$$

Substitution of the relation $\omega_2^{(2)} = -\Delta\psi_2^{(2)}$ into the left-hand side of (4.17) yields

$$D\psi_2^{(2)} = -(m^2/4\pi U^2)\,Q(\eta) + O(\xi^{-1}).$$

Putting

$$\psi_2^{(2)} = (m^2/4\pi U^2)\,g_2(\eta) + O(\xi^{-1}) \tag{4.18}$$

and dropping the terms of order higher than ξ^0, we have

$$g_2'' + 2\eta g_2' = -Q(\eta).$$

The general solution of this equation is easily found to be

$$\begin{aligned}g_2(\eta) = &-\tfrac{1}{2}(\eta^2 + \tfrac{9}{2})\,e^{-2\eta^2} - \tfrac{1}{2}\pi^{\frac{1}{2}}(\eta^3 + \tfrac{5}{2}\eta)\,e^{-\eta^2}\operatorname{erf}\eta\\ &-(2\pi)^{\frac{1}{2}}\,\eta\operatorname{erf} 2^{\frac{1}{2}}\eta + \tfrac{1}{2}C\pi^{\frac{1}{2}}\operatorname{erf}\eta,\end{aligned} \tag{4.19}$$

where C is a constant of integration. It should be noted that another constant of integration can be added to the right-hand side of (4.19), but this constant can be

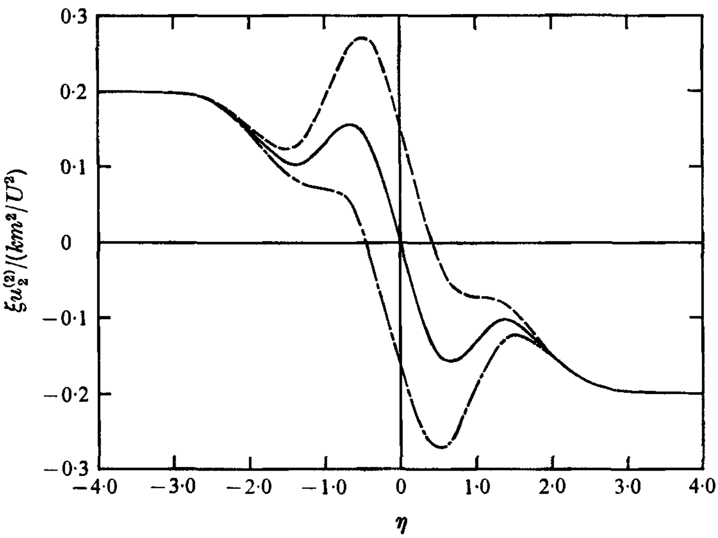

FIGURE 2. Streamwise velocity component in the second approximation. ---, $C = 2{\cdot}0$; ——, $C = 0$; — — —, $C = -2{\cdot}0$.

put equal to zero without loss of generality. Like the constants B and a in (3.8) and (3.10), respectively, C cannot be determined from the boundary conditions on the far wake. A tentative physical meaning of C will be discussed later.

The flow field outside the wake region will now be determined. Since

$$g_2(\eta) \to \mp (2\pi)^{\frac{1}{2}} (\eta - C/2^{\frac{3}{2}}) \quad \text{as} \quad \eta \to \pm\infty,$$

we have, from (4.18) and (4.19),

$$\psi_2^{(2)} \to \mp \frac{m^2}{2(2\pi)^{\frac{1}{2}} U^2} \left(\eta - \frac{C}{2^{\frac{3}{2}}}\right) \tag{4.20}$$

at the edge of the wake. Therefore, the flow outside the wake, which should be matched to (4.20), can be determined as

$$\psi_2^{(2)} = \mathscr{I}\{W_2^{(2)}\},$$

where

$$W_2^{(2)} = -\frac{m^2}{2(2\pi)^{\frac{1}{2}} U^2} \left\{(2kz)^{\frac{1}{2}} + \frac{C}{2^{\frac{3}{2}}\pi} \log z\right\}.$$

The velocity component in the x direction given by (4.18) and (4.19) is

$$u_2^{(2)} = \frac{\partial \psi_2^{(2)}}{\partial y} = \frac{km^2}{4\pi U^2} \xi^{-1} \{(\eta^3 + \tfrac{3}{2}\eta) e^{-2\eta^2} + \pi^{\frac{1}{2}} (\eta^4 + \eta^2 - \tfrac{5}{4}) e^{-\eta^2} \operatorname{erf} \eta$$
$$- (2\pi)^{\frac{1}{2}} \operatorname{erf} 2^{\frac{1}{2}} \eta + C e^{-\eta^2}\}, \tag{4.21}$$

which is plotted in figure 2 for a few values of C. It is worthwhile to note that the term $Ce^{-\eta^2}$ in the curly brackets introduces a symmetrical component into $u_2^{(2)}$, while all the remaining terms give the antisymmetrical profile.

4.3. *Third approximation*

The third approximation can be obtained in the same way as the second approximation. Write

$$\psi_2 = \psi_2^{(1)} + \psi_2^{(2)} + \psi_2^{(3)}. \tag{4.22}$$

By substituting (4.22) into (2.15) and retaining the terms of order $\xi^{-5}\log\xi$ and ξ^{-5}, we have

$$\begin{aligned}\Delta\left(\omega_2^{(3)} + 2k\frac{\partial\psi_2^{(3)}}{\partial x}\right) = {} & 2k\left(y\frac{\partial\omega_1^{(3)}}{\partial x} + \frac{\partial\psi_1^{(1)}}{\partial y}\frac{\partial\omega_2^{(2)}}{\partial x} + \frac{\partial\psi_1^{(2)}}{\partial y}\frac{\partial\omega_2^{(1)}}{\partial x} + \frac{\partial\psi_2^{(1)}}{\partial y}\frac{\partial\omega_1^{(2)}}{\partial x}\right.\\ & + \frac{\partial\psi_2^{(2)}}{\partial y}\frac{\partial\omega_1^{(1)}}{\partial x} - \frac{\partial\psi_1^{(1)}}{\partial x}\frac{\partial\omega_2^{(2)}}{\partial y} - \frac{\partial\psi_1^{(2)}}{\partial x}\frac{\partial\omega_2^{(1)}}{\partial y} - \frac{\partial\psi_2^{(1)}}{\partial x}\frac{\partial\omega_1^{(2)}}{\partial y}\\ & \left. - \frac{\partial\psi_2^{(2)}}{\partial x}\frac{\partial\omega_1^{(1)}}{\partial y}\right) + \frac{k^4m^2}{4\pi^{\frac{3}{2}}U^2}\xi^{-5}(\eta Z''' - Z'') + o(\xi^{-5}),\end{aligned} \tag{4.23}$$

in which the last term on the right-hand side corresponds to the term in curly brackets in (4.16) in the second approximation. Substitution of the above results into the right-hand side of (4.23) yields, after straightforward but lengthy calculations,

$$\begin{aligned}\Delta^*\left(\omega_2^{(3)} + 2k\frac{\partial\psi_2^{(3)}}{\partial x}\right) = {} & \frac{k^2m^2}{\pi^{\frac{3}{2}}U^2}\xi^{-3}\{2\pi^{\frac{1}{2}}[(\eta^2E'')' + 2(\eta E')' - 2E'] + [(\eta Y'')' - Y'']\\ & + \tfrac{1}{4}[(\eta Z'')' - 2Z''] - \log\xi[(\eta^2 Y''')' + 2(\eta Y'')' - 2Y'']\}\\ & + \frac{k^3m^3}{24\pi U^3}\xi^{-3}\left\{[(\eta^2 Q_3'')' + 2(\eta Q_3')' - 2Q_3']\right.\\ & \left. - 2\left(\frac{3}{\pi}\right)^{\frac{1}{2}}[(\eta Y'')' - Y''] + 2\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\log\xi[(\eta^2 Y''')' + 2(\eta Y'')' - 2Y'']\right\}\\ & + \frac{k^3m^3}{2\pi^{\frac{3}{2}}U^3}\xi^{-3}(YN_1')' - \frac{k^3m^3C}{2\pi U^3}\xi^{-3}(YE')' + \frac{k^3m^3}{16\pi U^3}\xi^{-3}[(XZ)'''\\ & - 2(XZ'')'] - \frac{Bk}{U}\xi^{-3}[(\eta^2Y''')' + 2(\eta Y'')' - 2Y''] + o(\xi^{-3}),\end{aligned} \tag{4.24}$$

where we have put

$$E = \operatorname{erf}\eta,$$

$$\begin{aligned}Q_3 = {} & 4\times 3^{\frac{1}{2}}\eta\, e^{-\eta^2}R(\eta) - 14\times 3^{\frac{1}{2}}(\operatorname{erf} 3^{\frac{1}{2}}\eta - \operatorname{erf}\eta) + 12\times 2^{\frac{1}{2}}e^{-\eta^2}\operatorname{erf} 2^{\frac{1}{2}}\eta\\ & + 6\, e^{-2\eta^2}\operatorname{erf}\eta + 3\pi^{\frac{1}{2}}e^{-\eta^2}(\operatorname{erf}\eta)^2 - 12\times 3^{\frac{1}{2}}\operatorname{erf}\eta,\end{aligned}$$

$$N_1 = (\eta^2 + \tfrac{9}{2})\,e^{-2\eta^2} + \pi^{\frac{1}{2}}(\eta^3 + \tfrac{5}{2}\eta)\,e^{-\eta^2}\operatorname{erf}\eta + 2(2\pi)^{\frac{1}{2}}\eta\operatorname{erf} 2^{\frac{1}{2}}\eta.$$

Proceeding just as in the derivation of (4.17), we find

$$\begin{aligned}\omega_2^{(3)} + 2k\frac{\partial\psi_2^{(3)}}{\partial x} & = \iint\Delta^*\left(\omega_2^{(3)} + 2k\frac{\partial\psi_2^{(3)}}{\partial x}\right)d\eta\, d\eta\\ & = -\frac{k^2m^2}{\pi^{\frac{3}{2}}U^2}\xi^{-3}\{F_1(\eta) + \log\xi F_2(\eta)\}\\ & \quad + \frac{k^3m^3}{48\pi U^3}\xi^{-3}\left\{F_3(\eta) + 4\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\log\xi\, F_2(\eta)\right\}\\ & \quad - \frac{Bk}{U}\xi^{-3}F_2(\eta) - \frac{k^3m^3C}{2^{\frac{3}{2}}\pi U^3}\xi^{-3}\operatorname{erf} 2^{\frac{1}{2}}\eta + o(\xi^{-3}),\end{aligned} \tag{4.25}$$

with F_1, F_2 and F_3 given by

$$F_1(\eta) = (\eta^4+\eta^2+\tfrac{39}{4})\,e^{-\eta^2}+4\pi^{\frac{1}{2}}\eta\,\mathrm{erf}\,\eta,$$

$$F_2(\eta) = 2(2\eta^4-\eta^2-1)\,e^{-\eta^2},$$

$$\begin{aligned}F_3(\eta) = {} & 8\times 3^{\frac{1}{2}}(1+\eta^2-2\eta^4)\,e^{-\eta^2}R(\eta)+8\times 3^{\frac{1}{2}}\eta^3(\mathrm{erf}\,3^{\frac{1}{2}}\eta-\mathrm{erf}\,\eta)\\ & +8(3/\pi)^{\frac{1}{2}}(2\eta^2+1)\,e^{-\eta^2}+(6/\pi^{\frac{1}{2}})(21-2\eta^2)\,e^{-3\eta^2}\\ & +18\times 2^{\frac{1}{2}}(\eta-2\eta^3)\,e^{-\eta^2}\,\mathrm{erf}\,2^{\frac{1}{2}}\eta+60\,\eta\,e^{-2\eta^2}\,\mathrm{erf}\,\eta\\ & +6\pi^{\frac{1}{2}}(1+\eta^2-2\eta^4)\,e^{-\eta^2}(\mathrm{erf}\,\eta)^2\\ & +48\times 3^{\frac{1}{2}}\eta\,\mathrm{erf}\,3^{\frac{1}{2}}\eta-36\int_0^\eta e^{-2\eta^2}\,\mathrm{erf}\,\eta\,d\eta.\end{aligned}$$

In view of the relation $\omega_2^{(3)} = -\Delta\psi_2^{(3)}$, equation (4.25) can be reduced to

$$\begin{aligned}D\psi_2^{(3)} = {} & \frac{m^2}{\pi^{\frac{3}{2}}U^2}\xi^{-1}\{F_1(\eta)+\log\xi F_2(\eta)\}-\frac{km^3}{48\pi U^3}\xi^{-1}\left\{F_3(\eta)+4\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\log\xi\,F_2(\eta)\right\}\\ & +\frac{B}{kU}\xi^{-1}F_2(\eta)+\frac{km^3C}{2^{\frac{3}{2}}\pi U^3}\xi^{-1}\,\mathrm{erf}\,2^{\frac{1}{2}}\eta.\end{aligned} \tag{4.26}$$

To find an appropriate solution of (4.26), we first consider the equation

$$D\psi_2^{(3)} = (m^2/\pi^{\frac{3}{2}}U^2)\,\xi^{-1}\{F_1(\eta)+\log\xi\,F_2(\eta)\}. \tag{4.27}$$

Assuming
$$\psi_2^{(3)} = (m^2/\pi^{\frac{3}{2}}U^2)\,\xi^{-1}\{g_{31}(\eta)+\log\xi\,g_{32}(\eta)\}, \tag{4.28}$$

and dropping the terms of order higher than ξ^{-1}, we have

$$g''_{31}+2\eta g'_{31}+2g_{31} = F_1+2g_{32}, \tag{4.29}$$

$$g''_{32}+2\eta g'_{32}+2g_{32} = F_2. \tag{4.30}$$

The general solution of (4.30) is

$$g_{32}(\eta) = -(\tfrac{1}{2}\eta^4+\eta^2)\,e^{-\eta^2}+C_{31}e^{-\eta^2}\int_0^\eta e^{\eta^2}d\eta+C_{32}e^{-\eta^2}, \tag{4.31}$$

where C_{31} and C_{32} are constants of integration. The second term on the right-hand side of (4.31) obeys

$$C_{31}e^{-\eta^2}\int_0^\eta e^{\eta^2}d\eta \sim C_{31}(2\eta)^{-1}\{1+(2\eta^2)^{-1}+O(\eta^{-4})\}$$

as $\eta\to\pm\infty$. Thus, this term gives rise to a term $C_{31}(2\xi\eta)^{-1} = C_{31}(2ky)^{-1}$ in $\psi_2^{(3)}$, which does not satisfy the requirement that the vorticity in the wake should exponentially tend to that in the flow outside the wake as the edge of the wake is approached (the principle of rapid decay of vorticity; Chang 1961). Accordingly, we put

$$C_{31} = 0. \tag{4.32}$$

Another constant C_{32} remains to be determined. Substituting (4.31) with (4.32) into (4.29) and integrating the resulting equation, we obtain

$$g_{31}(\eta) = \tfrac{1}{4}\eta^2e^{-\eta^2}+\pi^{\frac{1}{2}}\eta\,\mathrm{erf}\,\eta+\pi^{\frac{1}{2}}(C_{32}+\tfrac{21}{8})\,e^{-\eta^2}\int_0^\eta e^{\eta^2}\,\mathrm{erf}\,\eta\,d\eta+C_{33}e^{-\eta^2}\int_0^\eta e^{\eta^2}\,d\eta+C_{34},$$

C_{33} and C_{34} being constants of integration. Since the term

$$e^{-\eta^2}\int_0^\eta e^{\eta^2}\operatorname{erf}\eta\,d\eta,$$

in the same manner as the term

$$e^{-\eta^2}\int_0^\eta e^{\eta^2}\,d\eta,$$

does not satisfy the requirement of the principle of rapid decay of vorticity, we obtain

$$C_{32} = -\tfrac{21}{8}, \quad C_{33} = 0.$$

Therefore, the solutions of (4.29) and (4.30) can be determined as

$$g_{31}(\eta) = G_{31}(\eta) + C_1 e^{-\eta^2}, \tag{4.33}$$

$$g_{32}(\eta) = G_{32}(\eta), \tag{4.34}$$

in which C_1 is an arbitrary constant and

$$G_{31}(\eta) = \tfrac{1}{4}\eta^2 e^{-\eta^2} + \pi^{\frac{1}{2}}\eta\operatorname{erf}\eta, \tag{4.35}$$

$$G_{32}(\eta) = -\tfrac{1}{2}\eta^2(\eta^2 + \tfrac{29}{4})\,e^{-\eta^2}. \tag{4.36}$$

Next, consider the equation

$$D\psi_2^{(3)} = -(km^3/48\pi U^3)\,\xi^{-1}\{F_3(\eta) + 4(3/\pi)^{\frac{1}{2}}\log\xi F_2(\eta)\}. \tag{4.37}$$

Assuming
$$\psi_2^{(3)} = -(km^3/48\pi U^3)\,\xi^{-1}\{g_{33}(\eta) + \log\xi g_{34}(\eta)\}, \tag{4.38}$$

we have
$$g''_{33} + 2\eta g'_{33} + 2g_{33} = F_3 + 2g_{34},$$

$$g''_{34} + 2\eta g'_{34} + 2g_{34} = 4(3/\pi)^{\frac{1}{2}}F_2.$$

These equations can be solved in the same manner as (4.29) and (4.30). The results are

$$g_{33}(\eta) = G_{33}(\eta) + C_2 e^{-\eta^2}, \tag{4.39}$$

$$g_{34}(\eta) = G_{34}(\eta), \tag{4.40}$$

where C_2 is an arbitrary constant and

$$\begin{aligned} G_{33}(\eta) = {} & 2\times 3^{\frac{1}{2}}(\eta^4 + 2\eta^2 + 2)\,e^{-\eta^2}R(\eta) - 3^{\frac{1}{2}}(\eta^3 + \tfrac{5}{2}\eta)(\operatorname{erf}3^{\frac{1}{2}}\eta - \operatorname{erf}\eta) \\ & + (3/\pi^{\frac{1}{2}})(2\eta^2 + 7)\,e^{-3\eta^2} - (3/\pi)^{\frac{1}{2}}\eta^2 e^{-\eta^2} + 12\times 3^{\frac{1}{2}}\eta\operatorname{erf}3^{\frac{1}{2}}\eta \\ & + 3\times 2^{\frac{1}{2}}(2\eta^3 + 3\eta)\,e^{-\eta^2}\operatorname{erf}2^{\frac{1}{2}}\eta + 3\eta(\eta^2 + \tfrac{5}{2})\,e^{-2\eta^2}\operatorname{erf}\eta \\ & + \tfrac{27}{2}e^{-\eta^2}\int_\eta^\infty e^{-t^2}\operatorname{erf}t\,dt + 3\pi^{\frac{1}{2}}\eta^2(\tfrac{1}{2}\eta^2 + 1)\,e^{-\eta^2}(\operatorname{erf}\eta)^2 \\ & + 36\,e^{-\eta^2}\int_0^\eta \{S(\eta) - S(+\infty)\operatorname{erf}\eta\}\,e^{\eta^2}\,d\eta, \end{aligned}$$

$$G_{34}(\eta) = -(3/\pi)^{\frac{1}{2}}(2\eta^4 + 4\eta^2 + \tfrac{3}{2})\,e^{-\eta^2} - (36/\pi^{\frac{1}{2}})\,e^{-\eta^2}S(+\infty),$$

$$S(\eta) = \int_0^\eta\int_{\eta'}^\infty e^{-2t^2}\operatorname{erf}t\,dt\,d\eta'.$$

It should be pointed out that $S(\eta)$ is an odd function of η.

Consider the equation

$$D\psi_2^{(3)} = (B/kU)\,\xi^{-1}F_2(\eta).$$

Putting $\psi_2^{(3)} = (B/kU)\,\xi^{-1}g_{35}(\eta)$, we have

$$g''_{35} + 2\eta g'_{35} + 2g_{35} = F_2(\eta).$$

Since g_{35} satisfies the same equation as g_{32}, the solution becomes, from (4.31),

$$g_{35}(\eta) = G_{35}(\eta) + C_3\,e^{-\eta^2}, \tag{4.41}$$

where C_3 is an arbitrary constant of integration and

$$G_{35}(\eta) = -\eta^2(\tfrac{1}{2}\eta^2+1)\,e^{-\eta^2}.$$

Finally, consider the equation

$$D\psi_2^{(3)} = (km^3C/2^{\frac{3}{2}}\pi U^3)\,\xi^{-1}\,\mathrm{erf}\,2^{\frac{1}{2}}\eta.$$

When the solution of this equation is assumed to be of the form

$$\psi_2^{(3)} = (km^3C/2^{\frac{3}{2}}\pi U^3)\,\xi^{-1}g_{36}(\eta),$$

g_{36} satisfies the equation

$$g''_{36} + 2\eta g'_{36} + 2g_{36} = \mathrm{erf}\,2^{\frac{1}{2}}\eta.$$

The solution is

$$g_{36}(\eta) = G_{36}(\eta) + C_4 e^{-\eta^2}, \tag{4.42}$$

where

$$G_{36}(\eta) = \tfrac{1}{2}\,\mathrm{erf}\,2^{\frac{1}{2}}\eta - 2^{-\frac{1}{2}}\,e^{-\eta^2}\,\mathrm{erf}\,\eta,$$

C_4 being arbitrary.

It should be remarked here that, in obtaining the solutions of g_{33}–g_{36}, the principle of rapid decay of vorticity has been applied to determine the constants of integration other than C_2, C_3 and C_4.

From the foregoing analysis the solution in the wake region in the third approximation is given by

$$\psi_2^{(3)} = \frac{m^2}{\pi^{\frac{3}{2}}U^2}\,\xi^{-1}\{G_{31}(\eta) + \log\xi\,G_{32}(\eta)\} - \frac{km^3}{48\pi U^3}\,\xi^{-1}\{G_{33}(\eta) + \log\xi G_{34}(\eta)\}$$
$$+\frac{B}{kU}\,\xi^{-1}G_{35}(\eta) + \frac{km^3C}{2^{\frac{3}{2}}\pi U^3}\,\xi^{-1}G_{36}(\eta) + C'\xi^{-1}e^{-\eta^2}, \tag{4.43}$$

where C' is an arbitrary constant which cannot be determined from the boundary conditions on the far wake in a uniform shear flow. It may also be noted that the term multiplied by the undetermined constant C' in the third approximation yields an antisymmetrical part of the velocity component $u_2^{(3)}$ ($= \partial\psi_2^{(3)}/\partial y$) in the same manner as the remaining terms other than that multiplied by C.

To find the flow outside the wake in the third approximation, we note that

$$G_{31} \to \pm\,\pi^{\frac{1}{2}}\eta, \quad G_{33} \to \pm\,12\times 3^{\frac{1}{2}}\eta, \quad G_{36} \to \pm\,\tfrac{1}{2} \quad \text{as} \quad \eta \to \pm\infty.$$

Then, $\psi_2^{(3)}$ behaves asymptotically according to

$$\psi_2^{(3)} \to \pm\left\{\left(\frac{m^2}{\pi U^2} - \frac{3^{\frac{1}{2}}km^3}{4\pi U^3}\right)\xi^{-1}\eta + \frac{km^3C}{2^{\frac{5}{2}}\pi U^3}\,\xi^{-1}\right\}. \tag{4.44}$$

Accordingly, the flow outside the wake which matches to (4.44) is determined as

$$\psi_2^{(3)} = \mathscr{I}\{W_2^{(3)}\},$$

where

$$W_2^{(3)} = \pm\left(\frac{m^2}{2\pi U^2} - \frac{3^{\frac{1}{2}}km^3}{8\pi U^3}\right)\log z + i\frac{km^3C}{2^{\frac{5}{2}}\pi U^3}(2kz)^{-\frac{1}{2}},$$

the $\pm$ sign standing for $\mathscr{I}(z) \gtrless 0$.

5. Discussion of results

From the above analysis, the stream function representing the flow of an incompressible viscous fluid past a symmetrical two-dimensional body located in a uniform shear flow has been obtained asymptotically both in and outside the wake region. Since the analysis is based on the assumption that the shear can be regarded as a perturbation on a uniform stream, the region where the present solution is valid must be clarified. Noting that the solution has been obtained by an iterative procedure which starts from the solution of the linearized form of (2.14), we have, as an approximate estimate,

$$|\partial\psi_1^{(1)}(x,0)/\partial y| \ll 1. \tag{5.1}$$

Since, from (3.5),

$$\partial\psi_1^{(1)}(x,0)/\partial y = -(km/\pi^{\frac{1}{2}}U)(2kx)^{-\frac{1}{2}},$$

the condition (5.1) yields

$$(2kx)^{\frac{1}{2}} \gg km/\pi^{\frac{1}{2}}U. \tag{5.2}$$

Moreover, as is inferred from (3.3), the far wake in a uniform stream is limited to the region in which η is of order unity. This fact implies that the far wake is a vaguely parabolic region described by $\theta = O\{(2kx)^{-\frac{1}{2}}\}$, which gives an estimate of the width of the wake δ as

$$\delta = O\{x(2kx)^{-\frac{1}{2}}\}. \tag{5.3}$$

In order that the shear can be treated as a perturbation on a uniform stream, the shear length defined by $U/G = \alpha^{-1}$ must be much larger than δ, i.e.

$$\delta \ll \alpha^{-1},$$

or

$$(2kx)^{\frac{1}{2}} \ll k/\alpha. \tag{5.4}$$

Combining (5.2) and (5.4), we finally obtain

$$km/\pi^{\frac{1}{2}}U \ll (2kx)^{\frac{1}{2}} \ll k/\alpha. \tag{5.5}$$

Here it should be added that (5.5) is applied both in and outside the far-wake region.

The asymptotic solution described in the present analysis contains four constants a, B, C and C' which cannot be determined from the boundary conditions for the far wake. As was initially demonstrated by Stewartson (1957), these constants will be connected in some way with the flow field in the vicinity of the wake-producing obstacle, which is neglected in the asymptotic analysis. Mathematically, the terms which are multiplied by B, C and C', respectively, are the solutions of $(\Delta - 2k\partial/\partial x)\psi = 0$, while the term multiplied by a is the solution of $\Delta\psi = 0$. It should here be pointed out that the terms multiplied by C introduce the symmetrical parts into the velocity components $u_2\,(=\partial\psi_2/\partial y)$, which represents the effect of the uniform shear of the main flow, although all the other terms up to the third approximation yield the antisymmetrical parts. As a result, the defect of flow rate in the wake, which can be defined as

$$q = \int_{-\infty}^{+\infty} (U + Ky - u)\,dy,$$

is written in the form

$$q = -U\{\psi_1(\xi, +\infty) - \psi_1(\xi, -\infty)\} - G\{\psi_2(\xi, +\infty) - \psi_2(\xi, -\infty)\}$$

$$= \left(m - \frac{Gm^2C}{4\pi^{\frac{1}{2}}U^2}\right) + \left(\frac{km^2}{(2\pi)^{\frac{1}{2}}U} - \frac{Gkm^3C}{2^{\frac{3}{2}}\pi U^3}\right)\frac{1}{\xi} + o(\xi^{-1}). \tag{5.6}$$

In the same manner, the momentum defect M in the wake can be evaluated as

$$M = \rho\int_{-\infty}^{+\infty} (U + Ky)(U + Ky - u)\,dy$$

$$= \rho mU - \rho Gm^2C/4\pi^{\frac{1}{2}}U^2 + O(\xi^{-1}), \tag{5.7}$$

which, in the limit $\xi \to \infty$, could be interpreted as the drag force acting on the body placed in the uniform shear flow. However, since there exists the upper limit on x given by (5.5) or, in terms of ξ,

$$m/2\pi^{\frac{1}{2}}\nu \ll \xi \ll U^2/\nu G,$$

beyond which the present analysis is not valid, we cannot take such a limit in a strict sense. Nevertheless, it is the authors' impression that the constant C will have some connexion with the drag force acting on a symmetrical obstacle in a uniform shear flow.

In order to show more clearly the characteristics of wakes in a uniform shear flow, the velocity profiles in the wakes will now be examined. Consider a symmetrical body with representative length l and define the drag coefficient C_D and Reynolds number Re by

$$C_D = D/\tfrac{1}{2}\rho U^2 l, \quad Re = Ul/\nu,$$

respectively. It should be remembered that C_D means the drag coefficient when the obstacle is located in a uniform stream $u_\infty = U$. Then, the velocity defect w in the wake, which is defined by

$$w = (U + Gy) - u = -\partial\psi/\partial y,$$

becomes

$$-w/U = \tfrac{1}{4}C_D Re^{\frac{1}{2}}\{(x/l)^{-\frac{1}{2}}g_1^{(1)}(\eta) + \tfrac{1}{4}C_D Re^{\frac{1}{2}}(x/l)^{-1}g_1^{(2)}(\eta) + O[(x/l)^{-\frac{3}{2}}\log(x/l)]\}$$
$$+ \tfrac{1}{2}C_D(Gl/U)\{g_2^{(1)}(\eta) + \tfrac{1}{4}C_D Re^{\frac{1}{2}}(x/l)^{-\frac{1}{2}}g_2^{(2)}(\eta) + O[(x/l)^{-1}\log(x/l)]\}, \tag{5.8}$$

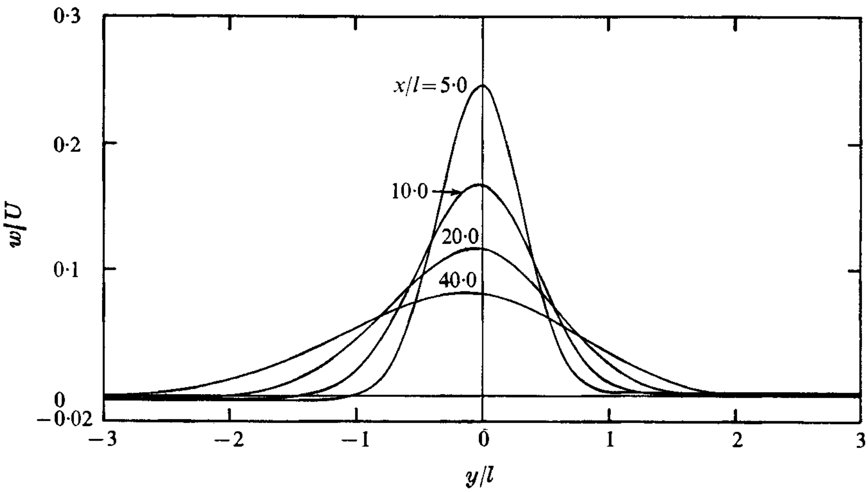

FIGURE 3. Velocity-defect profiles in the far wake of a flat plate of length l. $Gl/U = 0{\cdot}25$, $Re = 100$, $C = 0$.

where

$$g_1^{(1)}(\eta) = -\pi^{-\frac{1}{2}} e^{-\eta^2},$$
$$g_1^{(2)}(\eta) = -(2\pi^{\frac{1}{2}})^{-1} (\pi^{-\frac{1}{2}} e^{-\eta^2} + \eta e^{-\eta^2} \operatorname{erf} \eta),$$
$$g_2^{(1)}(\eta) = (4\pi^{\frac{1}{2}})^{-1} \eta (2\eta^2+3) e^{-\eta^2},$$
$$g_2^{(2)}(\eta) = (4\pi^{\frac{1}{2}})^{-1} \{(\eta^3 + \tfrac{3}{2}\eta) e^{-2\eta^2} + \pi^{\frac{1}{2}} (\eta^4 + \eta^2 - \tfrac{5}{4}) e^{-\eta^2} \operatorname{erf} \eta - (2\pi)^{\frac{1}{2}} \operatorname{erf} 2^{\frac{1}{2}}\eta + C e^{-\eta^2}\}.$$

In deriving (5.8), we have used the fact that, in the wake region ξ, almost satisfies

$$\xi = (2kx)^{\frac{1}{2}} = Re^{\frac{1}{2}}(x/l)^{\frac{1}{2}}.$$

Moreover, in terms of x/l, C_D and Re, the condition (5.5) can be rewritten as

$$C_D Re^{\frac{1}{2}}/4\pi^{\frac{1}{2}} \ll (x/l)^{\frac{1}{2}} \ll Re^{\frac{1}{2}}(U/Gl). \tag{5.9}$$

As an example, a finite flat plate of length l aligned parallel to the main flow will be considered here. The drag coefficient has been given by Kuo (1953) as

$$\tfrac{1}{2} C_D Re^{\frac{1}{2}} = 1{\cdot}328 + 4{\cdot}12 Re^{-\frac{1}{2}}. \tag{5.10}$$

The numerical calculations have been performed for the case $Re = 100$ and $Gl/U = 0{\cdot}25$ together with an arbitrarily chosen value of $C = 0$. In this case, (5.9) becomes approximately

$$0{\cdot}25 \ll (x/l)^{\frac{1}{2}} \ll 40.$$

The results are shown in figure 3, which gives the velocity-defect profiles at various sections downstream the plate. It can clearly be seen in the figure that the asymmetrical feature of the velocity defect becomes more evident as x increases. The width of the wake extends to the low-velocity side more than to the high-velocity side, and the point which shows the maximum velocity defect shifts to the low-velocity side.

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